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Six-photon amplitudes in scalar QED

C. Bernicot and J.-Ph. Guillet

LAPTH, Université de Savoie, CNRS,
B.P. 110, F-74941 Annecy-le-Vieux Cedex, France
E-mail: bernicot@lapp.in2p3.fr, guillet@lapp.in2p3.fr

Abstract: The analytical result for the six-photon helicity amplitudes in scalar QED is presented. To compute the loop, a recently developed method based on multiple cuts is used. The amplitudes for QED and $QED^{\mathcal{N}=1}$ are also derived using the supersymmetric decomposition linking the three theories.

Keywords: Supersymmetric Standard Model, Electromagnetic Processes and Properties, NLO Computations.

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1. Introduction and notation

The light-by-light scattering is a prediction of quantum electrodynamic despite the fact that it has never been observed so far. The four-photon amplitudes, in QED, have been computed in the fifties at one loop for massive fermion [1] and recently at two loops for massless fermions in QED [2] and in $\mathcal{N}=1$ supersymmetric QED [3]. The first result for the six-photon amplitudes, at one loop in QED and for massless fermions, was obtained for the MHV (Maximal Helicity Violating) amplitude by Mahlon [4]. The complete helicity amplitudes, in QED, has been computed numerically by direct integration of the Feynman diagrams [5], and also, by using reduction at the integrand level [6]. In the same time, it has also been computed analytically using unitary cut methods and cross checked with a reduction method [7]. A compact formula has been given, proving the power of the unitary cut methods. The unitarity-cut methods were developed first in [8] and then in [9]. They were used to simplify the computation of the processes $e^+e^- \to$ 4partons in [10]. They are currently under intense developments [11 − 15].

Although the six-photon amplitudes are out of reach for nowadays experiments, they provide a good laboratory reaction to settle efficient methods to compute one loop multi-leg amplitudes. Indeed, multi-particle processes involving Quantum ChromoDynamics (QCD) will play an important role in the physics probed by the hadronic colliders at the TeV scale. In particular at the future Large Hadron Collider (LHC), the production of four, and even five or six jets will not be marginal. Besides providing a refined probe of the dynamics of colour, such QCD processes constitute a background to the search for new particles. Indeed, the search for many of these new particles at hadronic colliders often relies on signatures based on cascade decays. The latter end up with final states involving a large jet multiplicity. Furthermore, the lowest order estimates for such processes are plagued by the well-known deficiencies of large renormalization and factorization scale dependencies, poor multi-jet modelling and large sensitivity to kinematic cuts. Therefore the calculation of next-to-leading-order (NLO) corrections to such cross sections is a necessary step forward.

In this article, for scalar QED (respectively QED, $\mathcal{N}=1$ supersymmetric QED) a generic N-photon helicity amplitude is denoted by: A_N^{scalar} (respectively A_N^{fermion}, $A_N^{\mathcal{N}=1}$). We will use unitarity-cut methods to compute the six-photon amplitudes in scalar QED. From these results, we can derive results for QED and $\mathcal{N}=1$ supersymmetric QED. To achieve this, we use a relation which relates the three theories. To find it, we proceed as follows. Starting with the N-photon QED amplitudes and using the fact that degrees of freedom for internal lines can be added and subtracted [16], we can write the following relation:

$$A_N^{\text{fermion}} = -2A_N^{\text{scalar}} + A_N^{\mathcal{N}=1} \tag{1.1}$$

To calculate those amplitudes, we use the spinor helicity formalism developed in [17]. For the spinorial product, we introduce the following notation:

$$\langle p_a - | p_b + \rangle = \langle ab \rangle \tag{1.2}$$

$$\langle p_a + | p_b - \rangle = [ab] \tag{1.3}$$

$$\langle p_a - | \not{p}_b | p_c - \rangle = \langle abc \rangle = [cba] = \langle p_c + | \not{p}_b | p_a + \rangle \tag{1.4}$$

$$\langle p_a + | \not{p}_b \, \not{p}_c | p_d - \rangle = [abcd] = -[dcba] = -\langle p_d + | \not{p}_c \, \not{p}_b | p_a - \rangle \tag{1.5}$$

Moreover we use $p_{i\ldots j} = p_i + \cdots + p_j$ and $s_{i\ldots j} = (p_i + \cdots + p_j)^2 = p_{i\ldots j}^2$. The outline of the paper is as follows. In section 2, we analyse the structure of the amplitudes, and compute the different tree amplitudes necessary for our calculation. In section 3, we give an analytical result for the six-photon amplitudes in scalar QED and in section 4, we derive analytical results for QED and $\mathcal{N} = 1$ supersymmetric QED. In section 5, we plot the different amplitudes for some kinematics and discuss potential problems.

2. Structure of the amplitudes

2.1 Decomposition of the amplitudes A_6^{fermion}, A_6^{scalar} and $A_6^{\mathcal{N}=1}$

The standard reduction methods, for example [18 – 20] show that any amplitude can be written as a combination of master integrals. This set of master integrals is not unique.

From now, we consider theory with massless particles. In the case of the six-photon amplitudes, we use the following decomposition:

$$A_6 = \sum_{i \in \sigma(1,2,3,4,5,6)} \Big(a_i \, F_4 + b_i \, F_3 + d_i \, F_{2A} + c_i \, F_{2B} + e_i \, F_1 + f_i \, F_0 + g_i \, I_3^{n3mass} + h_i \, I_3^{n2mass} + i_i \, I_3^{n1mass} + j_i \, I_2^n + \text{rational terms} \Big) \tag{2.1}$$

where F_4 (respectively F_3, F_1 and F_0) is the so called "finite part" of the four point function, in n dimensions, with 4 external masses (respectively three external masses, one external mass and zero external mass), F_{2A} (respectively F_{2B}) the so called "finite" part of the n dimensional four point function with two adjacent external masses (respectively with two opposite external masses). Only this set of functions $\{F_{2A}, F_{2B}, F_1\}$ will be used, their exact definition can be found, for example in [21], and to be self consistent we recall them in the appendix A. In addition, I_3^{n3mass} (respectively I_3^{n2mass}, I_3^{n1mass}) is the n dimensional three point function with three external masses (respectively two external masses, one external mass) and I_2^n is the n dimensional two point function. The IR divergences are carried by the function I_3^{n2m} and I_3^{n1m} and the UV one by the function I_2^n. Using unitary-cut methods, we only have to compute the coefficients $a_i \ldots j_i$ and rational terms. Most of them are related by Bose symmetry or parity.

2.2 Tree amplitudes

In the framework of unitary-cut methods, we need to compute first tree amplitudes. In this subsection, we will present only the tree amplitudes in QED and scalar QED useful for our six-photon amplitude computation. The needed tree amplitudes, are the amplitudes corresponding to the reactions: two scalars (fermions) into N-photons with the same helicity

Figure 1: Tree amplitudes needed for the six-photon amplitudes. The particles associated with a plain line are scalars or fermions.

and two scalars (fermions) into N-photons with the same helicity but one. We assume that all the photons are ingoing, and p_a and p_b are the four momentum of the scalars (fermions):

$$A^{\text{scalar}}_{\text{tree}}(1^+,\ldots,N^+) = 0 \tag{2.2}$$

$$A^{\text{fermion}}_{\text{tree}}(1^+,\ldots,N^+) = 0 \tag{2.3}$$

$$\begin{aligned}A^{\text{scalar}}_{\text{tree}}(1^+,\ldots,N^+,k^-) &= i\left(e\sqrt{2}\right)^{N+1} \sum_{\sigma(\{1\ldots N\}\backslash k)} \frac{1}{\langle 12\rangle\langle 23\rangle\ldots\langle N-1N\rangle}\frac{\langle ka1\rangle}{\langle 1a1\rangle}\frac{\langle kbN\rangle}{\langle NbN\rangle}\\ &= i\left(e\sqrt{2}\right)^{N+1}\frac{\langle ka\rangle\langle kb\rangle}{\prod_{j=1,j\neq k}^{N}\langle ja\rangle\langle jb\rangle}\langle ab\rangle^{N-1}\\ &= ie\sqrt{2}\frac{\langle ka\rangle\langle kb\rangle}{\langle ab\rangle}\prod_{i=1,i\neq k}^{N} S_i\end{aligned} \tag{2.4}$$

$$A^{\text{fermion}}_{\text{tree}}(1^+,\ldots,N^+,k^-) = A^{\text{scalar}}_{\text{tree}}(1^+,\ldots,N^+,k^-)\left(\frac{\langle ka\rangle}{\langle kb\rangle}+\frac{\langle kb\rangle}{\langle ka\rangle}\right) \tag{2.5}$$

where $S_i = -e\sqrt{2}\frac{\langle ab\rangle}{\langle ai\rangle\langle ib\rangle}$ is the eikonal factor. Note that in equation (2.5), a sum over the helicities of the fermions has been performed.

2.3 Additional reductions

Now using the properties of the QED theories, we can simplify furthermore the decomposition (2.1)of the six-photon amplitudes. Those comments and rules is available only for these amplitudes.

Remark 1. *Only the functions I_3^{n2m}, I_3^{n1m} are IR divergent. Since each diagram is not IR divergent, the coefficients h_i and i_i are zero.*

Each Feynman diagram of the six-photon amplitudes is free of IR divergences thanks to the numerator of the fermionic propagator in QED or the structure of the vertex in scalar QED. If the reduction is done by pinching propagators, we get sub-diagrams which are not IR divergent. After the reduction, we obtain three point sub-diagrams and the "finite" part of four point scalar integrals. Since the three point sub-diagrams are free of IR divergences, they cannot be expressed in term of one mass/two mass three point scalar integrals I_3^{n2m}, I_3^{n1m} and so the coefficients h_i and i_i are zero [20].

Remark 2. *Using standard reduction (for example [20]), we can show for QED and scalar QED, that the coefficients in front of two point functions are zero and also that the rational terms are zero.*

The first statement is in accordance with the fact that each diagram of the six-photon amplitudes is free of UV divergences. From that, it is not obvious that the different coefficients are zero, we have proven it by explicit calculation. As a consequence of remarks (1) and (2), there are no logarithmic terms in the six-photon amplitudes. The second statement, shown in ref. [22], is also far from being obvious. Indeed, according to power counting arguments [23], the rational terms can be present. In fact, they are present for individual Feynman diagram but these rational terms sum up to zero when adding all the diagrams.

Now, from what has been said previously, we can reduce the decomposition of the amplitude (2.1): for the six-photon case, the coefficients h_i, i_i, and j_i are zero and the rational term is also nul. In addition, since we have only six photons on shell, the coefficients a_i, b_i and f_i are also zero. So each amplitude A_6^{fermion}, A_6^{scalar} can be written as:

$$A_6 = \sum_{i \in \sigma(1,2,3,4,5,6)} d_i\, F_{2A} + c_i\, F_{2B} + e_i\, F_1 + g_i\, I_3^{n3mass} \tag{2.6}$$

From the helicity structure of trees, we can derive some rules which will reduce furthermore the decomposition (2.6).

Rule 1. *Consider a master integral with mass. If the mass is formed only with photons with the same helicity, therefore the coefficient in front of the master integral is zero.*

Proof: Using the cut techniques, we get that the coefficient in front of this integral is proportional to the tree amplitude which, once pinched, yields the mass. Formulae (2.2), (2.3) show that the on-shell tree amplitudes with photons having the same helicity are zero. So the coefficient of a master integral with a mass formed by photons with the same helicity is zero.

Rule 2. *For the box with one mass and with two adjacent masses, the helicity of two adjacent massless legs must be alternate. In the case of the box with two opposite masses, the helicities of the two opposite massless legs must be the same. If it is not the case, the coefficient in front of the master integral is zero.*

Proof: In the case of the one mass box and the two adjacent mass box, if the helicities of two adjacent massless legs are the same, the coefficient, in front of the box, will be proportional to trees, eqs. (2.2), (2.3), which are zero. The case of the two opposite mass box is more complicated and a proof is given in the appendix B.

Now we can further reduce the decomposition of all helicity amplitudes thanks to these last rules. We begin with the most simple amplitudes: $A_N(1^{\pm}, 2^{+} \ldots, N^{+})$. With at most one negative photon, we can have only one mass according to the rule (1), so $c_i, d_i, g_i = 0$. But in this case, the helicities cannot be alternate so, from the rule (2), we deduce that each $e_i = 0$. Therefore, we get for these amplitudes:

$$\forall N > 4, \qquad A_N^{\mathrm{scalar}}(1^\pm, 2^+ \ldots, N^+) = 0 \tag{2.7}$$
$$\forall N > 4, \qquad A_N^{\mathrm{fermion}}(1^\pm, 2^+ \ldots, N^+) = 0 \tag{2.8}$$
$$\forall N > 4, \qquad A_N^{\mathcal{N}=1}(1^\pm, 2^+ \ldots, N^+) = 0 \tag{2.9}$$

These results were already found by Mahlon [4] many years ago.

Then we study the MHV amplitude $A_6(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$. We have only two negative photons so we can only have at most two masses: $g_i = 0$. The helicities must be alternate according to rule (2) therefore for the MHV amplitude, $d_i = 0$. So we have:

$$A_6(--++++) = i\frac{(e\sqrt{2})^6}{16\pi^2} \sum_{\sigma(1,2)} \sum_{\sigma(3,4,5,6)} \frac{c_i}{2} F_{2B}(s_{215}, s_{415}, s_{15}, s_{26}) + \frac{e_i}{2} F_1(s_{23}, s_{24}, s_{156}) \tag{2.10}$$

Some permutations leave master integrals invariants, that is why we divide the coefficient by the adequate number.

It is rather easy to show that, in the case of the MHV six-photon amplitude, the two coefficients d_i and e_i are related. Indeed, we can take the limit such that the photon with helicity "+" forming one of the mass of the two opposite mass box is soft. In this limit, we have the following relation: $\lim_{m_2 \to 0} F_{2B} = F_1$. Since the MHV amplitude is composed by two MHV trees already expressed in term of eikonal factors eq. (2.4) and since the five photon amplitudes are zero, we can deduce that $c_i = -e_i$. Finally, we have the following decomposition for the MHV amplitude:

$$A_6(--++++) = i\frac{(e\sqrt{2})^6}{16\pi^2} \sum_{\sigma(1,2)} \sum_{\sigma(3,4,5,6)} \frac{c_i}{2} \left(F_{2B}(s_{315}, s_{415}, s_{15}, s_{26}) - F_1(s_{23}, s_{24}, s_{156})\right) \tag{2.11}$$

We have to compute only one coefficient.

Lastly, we examine the form of the decomposition of the Next to MHV amplitude (NMHV) $A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$. Here we can have three mass three point functions. But the rule (2) imposes $c_i = 0$. So we have only three coefficients to calculate and the amplitude can be written as:

$$\begin{aligned} A_6(---+++) = i\frac{(e\sqrt{2})^6}{16\pi^2} \sum_{\sigma(1,2,3)} \sum_{\sigma(4,5,6)} & \; 2\, d_i F_{2A}(s_{14}, s_{452}, s_{25}, s_{36}) + \frac{e_i}{2} F_1(s_{63}, s_{61}, s_{425}) \\ & + \frac{{e_i}^*}{2} F_1(s_{25}, s_{24}, s_{136}) + \frac{g_i}{6} I_3^{n3m}(s_{14}, s_{25}, s_{36}) \end{aligned} \tag{2.12}$$

Here again, we divide the coefficients by the adequate number to take into account permutations leaving invariant the corresponding master integrals. There are two kinds of one mass box integrals. The first kind has two photons with a positive helicity and one with a negative helicity forming the mass, whereas the second kind has one photon with a positive helicity and two photons with a negative helicity forming the mass. As we have three photons with a negative helicity and three with a positive helicity, the two kinds of one mass box integrals are directly related by parity. That is why the coefficient in front of them are complex conjugate. In the next section, we will give the results for the various cases.

3. A_6^{scalar} amplitudes

3.1 $A_N^{\text{scalar}}(1^-, 2^-, 3^+ \ldots, N^+)$, $N > 4$ helicity amplitudes

In this section, we calculate the MHV amplitude $A_N^{\text{scalar}}(1^-, 2^-, 3^+ \ldots, N^+)$ for N photons with $N > 4$. The generalization to N photons is rather easy, but note that we have to consider $N > 4$ otherwise there are some UV problems. Using the quadruple cut techniques [9], by direct computation, we obtain:

$$\begin{aligned} A_N^{\text{scalar}}(--+\cdots+) = \; & i\frac{\left(e\sqrt{2}\right)^N}{16\pi^2} \sum_{\sigma(1,2)} \sum_{\sigma(3\ldots N)} \frac{d_i^{\text{scalar}}}{(N-4)!} F_1(s_{23}, s_{24}, s_{15\ldots N}) \\ & + i\frac{\left(e\sqrt{2}\right)^N}{16\pi^2} \sum_{\sigma(1,2)} \sum_{\sigma(3\ldots N)} \sum_{M=5}^{N-1} \frac{(-1)^{M-6} d_i^{\text{scalar}}}{(N-M)!(M-4)!} F_{2B}(s_{135\ldots M}, s_{145\ldots M}, s_{15\ldots M}, s_{2M+1\ldots N}), \end{aligned} \tag{3.1}$$

where

$$d_i^{\text{scalar}} = -\frac{\langle 34\rangle^{N-6}}{\prod_{j=5}^{N}\langle 3j\rangle\langle 4j\rangle} \frac{\langle 13\rangle\langle 41\rangle\langle 23\rangle\langle 42\rangle}{s_{34}} \frac{[34]}{\langle 34\rangle} \tag{3.2}$$

The factorial coefficient $(N-M)!(M-4)!$ and $(N-4)!$ are the number of permutations which leaves invariant the mass of the master integral. The formula obtained is explicitly invariant by exchange of the two photons with a negative helicity. In the case where $N = 6$, the amplitude reduces to:

$$\begin{aligned} A_6^{\text{scalar}}(--++++) = -i\frac{\left(e\sqrt{2}\right)^6}{16\pi^2} \sum_{\sigma(1,2)} \sum_{\sigma(3\ldots 6)} \frac{\langle 13\rangle\langle 41\rangle\langle 23\rangle\langle 42\rangle}{\langle 35\rangle\langle 45\rangle\langle 36\rangle\langle 46\rangle} \left(\frac{[34]}{\langle 34\rangle} \frac{F_1(s_{23}, s_{24}, s_{156})}{s_{34}} \right. \\ \left. - \frac{F_{2B}(s_{135}, s_{145}, s_{15}, s_{26})}{s_{34}} \right) \end{aligned} \tag{3.3}$$

It remains only one Gram determinant (s_{34}) in the denominator but F_1, $F_{2B} \simeq s_{34}$ when $s_{34} \to 0$. Therefore the potential numerical problem when the Gram determinant vanishes is under control.

3.2 $A_6^{\text{scalar}}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ helicity amplitude

We use the quadruple-cut techniques to calculate the box coefficients [9], and the triple-cut technique to calculate the triangle coefficients [24]. We obtain for the amplitude:

$$
\begin{aligned}
A_6^{\rm scalar}(---+++) = i\frac{(e\sqrt{2})^6}{16\pi^2}\sum_{\sigma(1,2,3)}\sum_{\sigma(4,5,6)} & 2\ d_i^{\rm scalar}F_{2A}(s_{14},s_{452},s_{25},s_{36}) \\
& + \frac{e_i^{\rm scalar}}{2}F_1(s_{63},s_{61},s_{425}) \\
& + \frac{e_i^{\rm scalar\,*}}{2}\ F_1(s_{25},s_{24},s_{136}) \\
& + \frac{g_i^{\rm scalar}}{6}I_3^{n3m}(s_{14},s_{25},s_{36}) \qquad (3.4)
\end{aligned}
$$

where

$$d_i^{\rm scalar} = -\frac{s_{425}\langle 24\rangle[16]}{\langle 45\rangle[31][1p_{425}4]^2}\frac{[1p_{425}2][6p_{425}4]}{[1p_{425}5][3p_{425}4]} \qquad (3.5)$$

$$e_i^{\rm scalar} = -\frac{\langle 2p_{425}1\rangle\langle 2p_{425}3\rangle[36][16]s_{425}}{\langle 4p_{425}1\rangle\langle 5p_{425}3\rangle\langle 5p_{425}1\rangle\langle 4p_{425}3\rangle}\frac{\langle 31\rangle}{[31]s_{31}} \qquad (3.6)$$

$$e_i^{\rm scalar*} = -\frac{[6p_{613}5][6p_{613}4]\langle 42\rangle\langle 52\rangle s_{613}}{[1p_{613}4][1p_{613}5][3p_{613}4][3p_{613}5]}\frac{[45]}{\langle 45\rangle s_{54}} \qquad (3.7)$$

$$g_i^{\rm scalar} = \frac{[4p_{25}1]}{[1p_{25}4]}\frac{[5p_{14}2]}{[2p_{14}5]}\frac{[6p_{25}3]}{[3p_{25}6]}\sum_{\gamma_\pm}\frac{[1K_2^b1]}{[4K_2^b4]}\frac{[2K_2^b2]}{[5K_2^b5]}\frac{[3K_2^b3]}{[6K_2^b6]} \qquad (3.8)$$

$$K_2^{b\mu} = \gamma_\pm\,(-p_{25})^\mu - s_{25}\,(p_{14})^\mu \qquad (3.9)$$

$$\gamma_\pm = -p_{25}.p_{14} \pm \sqrt{\Delta} \qquad (3.10)$$

$$\Delta = (p_{25}.p_{14})^2 - p_{14}^2 p_{25}^2 \qquad (3.11)$$

The result is very compact. By expanding the coefficient $g_i^{\rm scalar}$, we find that all square roots, coming from $\gamma_\pm$, disappear as it should be. All the coefficients are rational functions of spinor products of external momenta. There is also a Gram determinant in the denominator of the coefficients $d_i^{\rm scalar}$ and $g_i^{\rm scalar}$. These two Gram determinants go to zero in the same phase space region. For this region, in the numerator, it is a combination of F_{2A} and I_3^{n3m} which cancels in such way that there is no singularity (for more details see [20]). We do not give here the explicit formulae because they break the simplicity of the expressions but we implement them in our numerical code. It is particulary important to do this for the scanning of the Landau singularities.

In the next section, we present the result for the amplitudes with a fermion loop and a sfermion loop.

4. $A_6^{\rm fermion}$ and $A_6^{\mathcal{N}=1}$ amplitudes

4.1 $A_N^{\rm fermion}(1^-,2^-,3^+\ldots,N^+)$ and $A_N^{\mathcal{N}=1}(1^-,2^-,3^+\ldots,N^+)$ helicity amplitudes

To calculate the MHV amplitudes: $A_N^{\rm fermion}(1^-,2^-,3^+\ldots,N^+)$ and $A_N^{\mathcal{N}=1}(1^-,2^-,3^+\ldots,N^+)$, we use extensively the formula (2.5) at the integrand level. The idea is the following. The introduction of (2.5) under the integral, in the cut

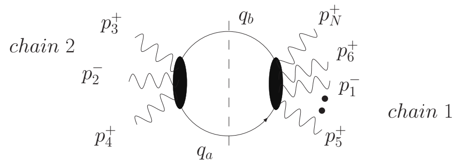

Figure 2: Spinor QED loop.

fermion amplitude ${A_N^{\text{fermion}}}_{\text{cut}}$, allows to write the cut fermion amplitude as the cut scalar amplitude ${A_N^{\text{scalar}}}_{\text{cut}}$ plus some terms. Using the supersymmetric decomposition (1.1) the remaining part is identified with the cut supersymmetric amplitude ${A_N^{\mathcal{N}=1}}_{\text{cut}}$. So, we do not have to calculate all the supersymmetric diagrams.

We show how it works on an example. We consider a fermion loop with two cuts on propagators q_a and q_b. The figure 2 shows how the helicities are shared: two trees with one photon with a negative helicity. We compute the cut amplitude ${A_N^{\text{fermion}}}_{\text{cut}}$ corresponding to figure 2:

$$\begin{aligned}
{A_N^{\text{fermion}}}_{\text{cut}} &= -\int d^n q\, \delta\left(q_a^2\right) \delta\left(q_b^2\right) \langle b\left(\Pi_+ + \Pi_-\right)(\text{chain1})a\rangle\langle a\left(\Pi_+ + \Pi_-\right)(\text{chain2})b\rangle \\
&= -\int d^n q\, \delta\left(q_a^2\right) \delta\left(q_b^2\right) \left(\frac{\langle 1b\rangle\langle 2a\rangle}{\langle 1a\rangle\langle 2b\rangle} + \frac{\langle 1a\rangle\langle 2b\rangle}{\langle 1b\rangle\langle 2a\rangle}\right) {A_1^{\text{scalar}}}_{\text{tree}} {A_2^{\text{scalar}}}_{\text{tree}} \\
&= -2\int d^n q\, \delta\left(q_a^2\right) \delta\left(q_b^2\right) {A_1^{\text{scalar}}}_{\text{tree}} {A_2^{\text{scalar}}}_{\text{tree}} \\
&\quad - \langle 12\rangle^2 \int d^n q\, \delta\left(q_a^2\right) \delta\left(q_b^2\right) \frac{s_{\text{ab}}^2 {A_1^{\text{scalar}}}_{\text{tree}} {A_2^{\text{scalar}}}_{\text{tree}}}{\langle 1ab1\rangle\langle 2ab2\rangle}
\end{aligned} \tag{4.1}$$

where $\Pi_\pm = \frac{1\pm\gamma_5}{2}$ are the chiral projectors. We recognize the cut scalar amplitude ${A_N^{\text{scalar}}}_{\text{cut}}$ in the first term of the right hand side of equation (4.1). Since the cut amplitude ${A_N^{\text{fermion}}}_{\text{cut}}$ also obeys to the supersymmetric decomposition (1.1), that means that the second term of the left hand side of eq. (4.1) is identified as the cut supersymmetric amplitude. Using this trick, we can easily calculate the supersymmetric amplitude and obtain straightforwardly the spinor amplitude. For these two amplitudes, we get:

$$\begin{aligned}
A_N^{fermion/\mathcal{N}=1}(--+\cdots+) = {}& i\frac{\left(e\sqrt{2}\right)^N}{16\pi^2} \sum_{\sigma(1,2)} \sum_{\sigma(3\ldots N)} \frac{d_i^{fermion/\mathcal{N}=1}}{(N-4)!} F_1(s_{23}, s_{24}, s_{15\ldots N}) \\
+ i\frac{\left(e\sqrt{2}\right)^N}{16\pi^2} & \sum_{\sigma(1,2)} \sum_{\sigma(3\ldots N)} \sum_{M=5}^{N-1} \frac{(-1)^{M-6} d_i^{fermion/\mathcal{N}=1}}{(N-M)!(M-4)!} F_{2B}(s_{135\ldots M}, s_{145\ldots M}, s_{5\ldots M}, s_{M+1\ldots N}),
\end{aligned} \tag{4.2}$$

where $d_i^{\text{fermion}} = 2\frac{\langle 34\rangle^{N-6}}{\prod_{j=5}^N \langle 3j\rangle\langle 4j\rangle} \frac{\langle 13\rangle^2\langle 42\rangle^2}{s_{34}} \frac{[34]}{\langle 34\rangle}$, and $d_i^{\mathcal{N}=1} = -\frac{\langle 34\rangle^{N-6}}{\prod_{j=5}^N \langle 3j\rangle\langle 4j\rangle} \langle 12\rangle^2$. In the case of

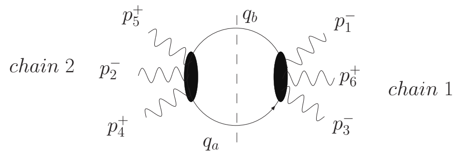

Figure 3: Spinor QED loop.

six photons, we get:

$$A_6^{\text{fermion}}(--++++) = i\frac{\left(e\sqrt{2}\right)^6}{16\pi^2}\sum_{\sigma(1,2)}\sum_{\sigma(3...6)}\frac{2\langle 13\rangle^2\langle 42\rangle^2}{\langle 35\rangle\langle 45\rangle\langle 36\rangle\langle 46\rangle}\frac{[34]}{\langle 34\rangle}\left(\frac{F_1(s_{23},s_{24},s_{156})}{s_{34}}\right.$$
$$\left.-\frac{F_{2B}(s_{135},s_{145},s_{15},s_{26})}{s_{34}}\right) \quad (4.3)$$
$$A_6^{\mathcal{N}=1}(--++++) = i\frac{\left(e\sqrt{2}\right)^6}{16\pi^2}\sum_{\sigma(1,2)}\sum_{\sigma(3...6)}\frac{-\langle 12\rangle^2}{\langle 35\rangle\langle 45\rangle\langle 36\rangle\langle 46\rangle}\left(F_1(s_{23},s_{24},s_{156})\right. \quad (4.4)$$
$$\left.-F_{2B}(s_{135},s_{145},s_{15},s_{26})\right)$$

Full agreement is found with [4, 7] for the fermion amplitude $A_6^{\text{fermion}}(--++++)$. We can do the same comments as in scalar QED. We point out that the Gram determinants s_{34} have disappeared in the supersymmetric amplitude $A_6^{\mathcal{N}=1}(--++++)$.

4.2 $A_6^{\text{fermion}}(1^-,2^-,3^-,4^+,5^+,6^+)$ and $A_6^{\mathcal{N}=1}(1^-,2^-,3^-,4^+,5^+,6^+)$ helicity amplitudes

To calculate those two amplitudes, we use again quadruple cut techniques [9] and triple cut techniques [24]. As in the preceding subsection, we use the supersymmetric decomposition (1.1) to extract the supersymmetric amplitude and then obtain the fermion one.

As in the previous subsection, we treat an example to illustrate how it works. It is a bit different here because the trees do not have the same helicity structure. We consider a fermion loop with two propagators cut as shown on figure 3. If we compute the corresponding cut amplitude, we get:

$$A_N^{\text{fermion}}{}_{\text{cut}} = -\int d^nq\,\delta\left(q_a^2\right)\delta\left(q_b^2\right)\langle b\left(\Pi_+ + \Pi_-\right)(\text{chain1})a\rangle\langle a\left(\Pi_+ + \Pi_-\right)(\text{chain2})b\rangle$$
$$= -2\int d^nq\,\delta\left(q_a^2\right)\delta\left(q_b^2\right)A_1^{\text{scalar}}{}_{\text{tree}}A_2^{\text{scalar}}{}_{\text{tree}}$$
$$-\langle 1P_{263}6\rangle^2\int d^nq\,\delta\left(q_a^2\right)\delta\left(q_b^2\right)\frac{A_1^{\text{scalar}}{}_{\text{tree}}A_2^{\text{scalar}}{}_{\text{tree}}}{\langle 1a6\rangle\langle 1b6\rangle} \quad (4.5)$$

Again, thanks to the supersymmetric decomposition (1.1), we can identify the supersymmetric cut amplitude and obtain the fermion cut amplitude. Doing all the calculation, we get for these two amplitudes:

$$A_6^{fermion/\mathcal{N}=1}(---+++)=i\frac{\left(e\sqrt{2}\right)^6}{16\pi^2}\sum_{\sigma(1,2,3)}\sum_{\sigma(4,5,6)}\Big(2\,d_i^{fermion/\mathcal{N}=1}F_{2A}(s_{14},s_{452},s_{25},s_{36})$$
$$+\frac{e_i^{fermion/\mathcal{N}=1}}{2}F_1(s_{63},s_{61},s_{425})+\frac{e_i^{fermion/\mathcal{N}=1\,*}}{2}F_1(s_{25},s_{24},s_{136})$$
$$+\frac{g_i^{fermion/\mathcal{N}=1}}{6}I_3^{n3m}(s_{14},s_{25},s_{36})\Big) \tag{4.6}$$

where:

$$d_i^{\text{fermion}} = -\frac{1}{[31]\langle 45\rangle[1p_{425}4]^2}\frac{[1p_{425}2]^2[6p_{425}4]^2+s_{425}^2\langle 24\rangle^2[16]^2}{[1p_{425}5][3p_{425}4]} \tag{4.7}$$

$$e_i^{\text{fermion}} = 2\frac{\langle 2p_{425}3\rangle^2[16]^2 s_{425}}{\langle 4p_{425}1\rangle\langle 5p_{425}3\rangle\langle 5p_{425}1\rangle\langle 4p_{425}3\rangle}\frac{\langle 31\rangle}{[31]s_{31}} \tag{4.8}$$

$$e_i^{\text{fermion}*} = 2\frac{[6p_{613}5]^2\langle 42\rangle^2 s_{613}}{[1p_{613}4][1p_{613}5][3p_{613}4][3p_{613}5]}\frac{[54]}{\langle 54\rangle s_{54}} \tag{4.9}$$

$$g_i^{\text{fermion}} = -2g_i^{\text{scalar}} \tag{4.10}$$

Again, we get a full agreement with [7] for the amplitude $A_6^{\text{fermion}}(---+++)$. We can observe a factor "2", except for the coefficient of the two adjacent box between the coefficients of the two amplitudes $A_6^{\text{fermion}}(---+++)$ and $A_6^{\text{scalar}}(---+++)$. This factor "2" comes from the fact that for a fermion loop, there are two currents. These two currents give rise to a factor "2" except in the case of the two adjacent mass box, where they give rise to a sum two terms (eq. (4.7)).

$$d_i^{\mathcal{N}=1} = -\frac{1}{[31]\langle 45\rangle}\frac{[6p_{425}2]^2}{[1p_{425}5][3p_{425}4]} \tag{4.11}$$

$$e_i^{\mathcal{N}=1} = 2d_i^{\mathcal{N}=1} \tag{4.12}$$

$$e_i^{\mathcal{N}=1*} = e_i^{\mathcal{N}=1} \tag{4.13}$$

$$g_i^{\mathcal{N}=1} = 0 \tag{4.14}$$

In the $\mathcal{N}=1$ supersymmetric QED, the coefficients are simpler. Since there are only six photons, the two kinds of one mass four point boxes are related by parity. This leads to the equality between the coefficients $e_i^{\mathcal{N}=1*}=e_i^{\mathcal{N}=1}$. With eight photons or more, this becomes wrong. We can also point out that all Gram determinants disappear.

4.3 Absence of triangles in $A_6^{\mathcal{N}=1}$

In the $\mathcal{N}=1$ supersymmetric amplitude $A_6^{\mathcal{N}=1}$, the fact that there are no triangles is probably just an accident. But we can make the following statement: for one loop, in $QED^{\mathcal{N}=1}$ theory, the N photons NMHV helicity amplitudes will have no triangle function. In fact, as we have only three photons with a negative helicity, each mass of the triangle is formed with one photon with a negative helicity. The relation (2.5) shows the linearity between the MHV tree amplitude in QED and in scalar QED. So we obtain directly using [24] that:

$$g^{\mathcal{N}=1} = g^{\text{fermions}} + 2g^{\text{scalar}} \tag{4.15}$$

But now if we have at least four photons with a negative helicity, one mass of the triangle will be formed by two photons with a negative helicity and the tree amplitude corresponding to this mass will not be a MHV tree. The problem comes from the fact that the relation linking NMHV tree amplitude in QED and scalar QED is not linear but affine. This affine coefficient is the contribution of the triangle to the supersymmetric $\mathcal{N} = 1$ amplitude and there is no reason that the coefficients in front of the triangles become zero. So we can conjecture that at one loop, in $QED^{\mathcal{N}=1}$ theory, the $N > 6$ photons Next to Next to MHV (NNMHV) helicity amplitudes will have triangles.

5. Numerical results

In this section, we present numerical results for the different six-photon amplitudes. We have built a fortran code using the GOLEM library for the different four point and three point functions. To be consistent with previous results, we use the kinematics defined by Nagy and Soper in ref. [5]. First of all, we recall this kinematics: the reaction $\gamma^{\lambda_1}(k_1) + \gamma^{\lambda_2}(k_2) \to \gamma^{\lambda_3}(k_3) + \gamma^{\lambda_4}(k_4) + \gamma^{\lambda_5}(k_5) + \gamma^{\lambda_6}(k_6)$ is considered, where k_i is the four-momentum of the photon i and λ_i its helicity, the four-momenta fulfill $k_1 + k_2 = k_3 + k_4 + k_5 + k_6$. In the center of mass frame $\vec{k}_1 + \vec{k}_2 = \vec{0}$, $\vec{k}_1$ is along the -z-axis, an arbitrary phase space point is chosen:

$$\begin{cases} \overrightarrow{k_3} = (33.5, 15.9, 25.0) \\ \overrightarrow{k_4} = (-12, 5, 15.3, 0.3) \\ \overrightarrow{k_5} = (-10.0, -18.0, -3.3) \\ \overrightarrow{k_6} = (-11.0, -13.2, -22.0) \end{cases} \tag{5.1}$$

Then new final momentum configurations is generated by rotating the final state through angle θ about the y-axis. For all the plots of this section, we take $\alpha = e^2/4\pi = 1$. The helicities "+" and "-" refer always to ingoing photons.

5.1 The MHV amplitudes.

In the figure 4, we plot the module of the MHV amplitude for QED, scalar QED and $\mathcal{N} = 1$ supersymmetric QED (respectively the formula (4.3) , (3.3) and (4.4)) against the variable θ. Note that these formula have been derived assuming that all the photons are ingoing. In order to match previous results of the references [5, 7], we compute $A_6^{fermion/scalar/\mathcal{N}=1}(k_1, k_2, -k_3, -k_4, -k_5, -k_6)$ with the helicities $\lambda_1 = -$, $\lambda_2 = -$, $\lambda_3 = +$, $\lambda_4 = +$, $\lambda_5 = +$, $\lambda_6 = +$.

All these MHV amplitudes are π periodic.

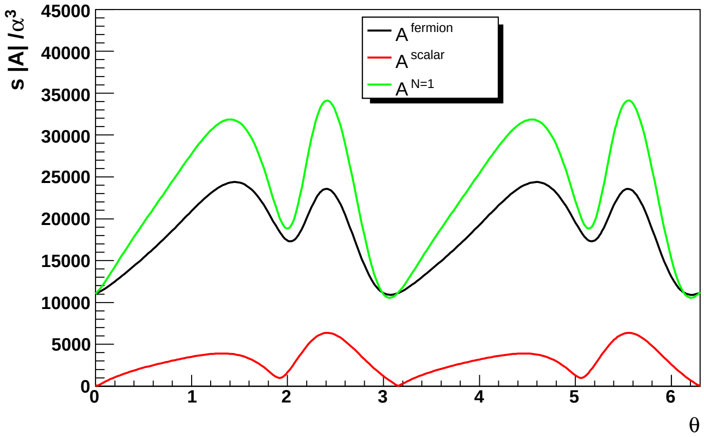

Figure 4: The MHV six-photon amplitudes with the Nagy and Soper [5] configuration for the three theories. Note that the curve for A^{scalar} amplitude has a minimum which is not zero.

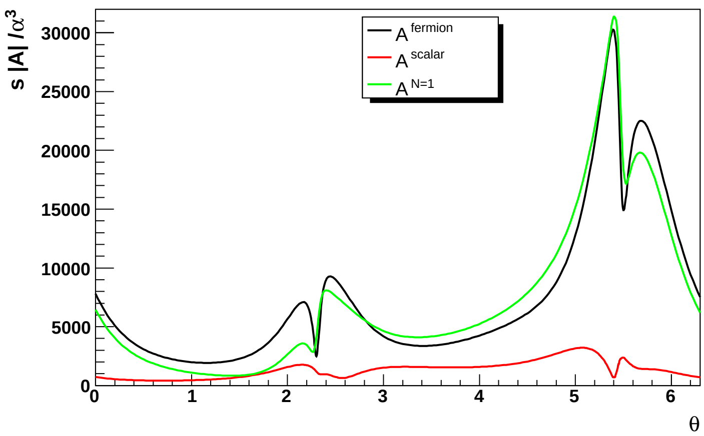

Figure 5: NMHV six-photon amplitudes with the Nagy and Soper [5] configuration for the three QED.

5.2 The NMHV amplitudes

In figure 5, with the same configuration than in the last section, we plot the NMHV six-photon amplitudes for the three theories against θ. We compute $A_6^{fermion/scalar/\mathcal{N}=1}(k_2, -k_3, -k_6, k_1, -k_4, -k_5)$ with the helicities $\lambda_1 = +$, $\lambda_2 = -$, $\lambda_3 = -$, $\lambda_4 = +$, $\lambda_5 = +$, $\lambda_6 = -$.

In this case, the amplitudes are not π periodic. The dips appearing in the curves, are related to the Landau singularities called the "double parton scattering" [5].

5.3 Double parton scattering

The Landau equations give the necessary conditions for a Feynman diagram to have a singularity. In the case of the six-photon amplitudes, since all internal and external particles are massless, three types of singularities can appear. Two are the well known soft and

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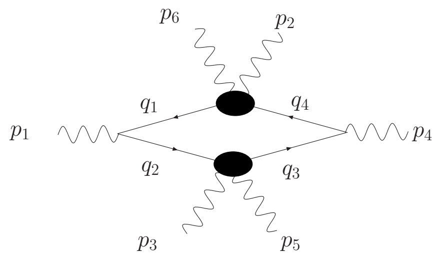


Figure 6: Double parton scattering configuration: p_1, p_4 are ingoing photons and p_2, p_3, p_5, p_6 are outgoing photons.

collinear singularities, the other corresponds to the so called double parton scattering. In this case, four propagators are one mass shell, these propagators are adjacent by pair.

So here we explain rapidly what is the "double parton scattering" kinematics.

The two ingoing photons 1 and 4 split into a fermion anti-fermion collinear pairs, each fermion scatter with an anti-fermion to give a photon pair with no transverse momentum in the center of mass frame $\vec{p}_1 + \vec{p}_4 = \vec{0}$ (c.f. figure 6). In the configuration of "double parton scattering", the two propagators q_1 and q_2 are collinear to the external leg p_1 and the two other q_3 and q_4 are collinear to p_4:

$$\begin{cases} q_1 = -x p_1 \\ q_2 = (1-x) p_1 \\ q_3 = -y p_4 \\ q_4 = (1-y) p_4 \end{cases} \tag{5.2}$$

Solving the Landau equations, we find that the conditions to have a double parton scattering singularity are:

$$\begin{cases} det(S) \to 0 \\ s_{35}, s_{26} > 0 \\ s_{135}, s_{435} < 0 \end{cases} \tag{5.3}$$

with

$$det(S) = s_{135} s_{435} - s_{35} s_{26} \tag{5.4}$$

In the center of mass frame $\vec{p}_1 + \vec{p}_4 = \vec{0}$, $det(S) = s_{14}\, k_t^2$ where k_t^2 is the square of the transverse momentum of the photon photon pairs 2,6 and 3,5.

In the figure 7, we plot the NMHV QED amplitudes as a function of θ. On the top of that, we surimpose k_t^2, normalized in such way that the curve is visible. We note that the dips appear in the region where k_t^2 is minimum. In this case, the minimum of k_t^2 is different from zero because we are not sitting on the singularity.

Choosing an appropriate kinematics containing a Landau singularity, we examined numerically the behavior of the six-photon amplitudes around this singularity. In all cases, the numerator goes to zero fast enough to compensate the cancellation of the denominator. Now we are studying analytically this compensation. More details will be presented in a forthcoming publication [25].

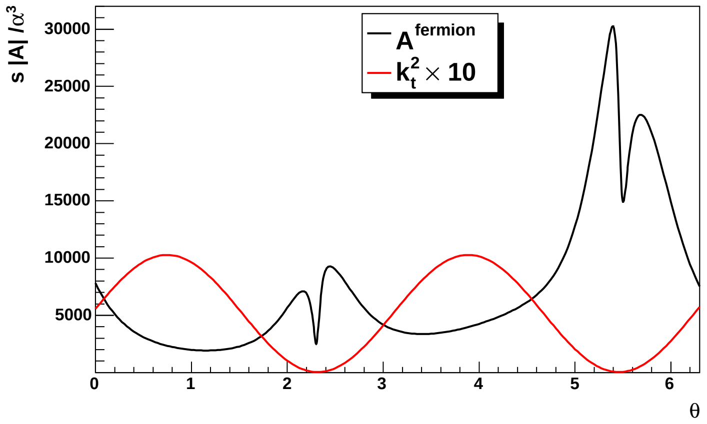


Figure 7: Localisation of a Landau singularity. Note that the red curve describing the k_t^2 of the photon pairs 2,6 and 3,5 does not reach zero.

6. Conclusion

In this paper, we have obtained all six-photon helicity amplitudes in QED, scalar QED and $\mathcal{N} = 1$ supersymmetric QED. Those amplitudes are linked among themselves by the relation (1.1). To calculate them, we used the powerful unitarity-cut techniques and we got very compact expressions. More work is required to understand quantitatively the behavior of these amplitudes and especially how the numerator regularize the double parton scattering singularity in a kinematics where it shows up, this will be presented elsewhere [25].

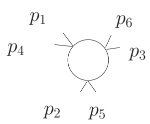
Acknowledgments

We want to thank T. Binoth, G. Heinrich and C. Schubert for collaboration during the early stages of this work and for useful discussions about ref. [7]. We also would like to thank E. Pilon and Ninh Le Duc for useful discussions on the Landau singularity and P. Aurenche for a careful reading of the manuscript.

A. Scalar integrals

In this appendix, for sake of completeness, the definition of master integrals used in this paper is recalled, more details can be found in [21]. We also give $det(G)$ the determinant of the Gram matrix $G_{ij} = 2p_i.p_j$ built with the external four momentum and $det(S)$ the determinant of the kinematical S-matrix defined by $S_{ij} = (q_j - q_i)^2$ where the q_i are the four momentum flowing in the propagators.

A.1 Three mass three point function



$$\begin{cases} m_1^2 = s_{14} \\ m_2^2 = s_{25} \\ m_3^2 = s_{36} \end{cases}$$

$$I_3^n\left(m_1^2, m_2^2, m_3^2\right) = \frac{1}{\sqrt{\Delta}}\left\{\left(2Li_2\left(1-\frac{1}{y_2}\right)+2Li_2\left(1-\frac{1}{x_2}\right)+\frac{\pi^2}{3}\right)\right.$$
$$\left.+\frac{1}{2}\left(\ln^2\left(\frac{x_1}{y_1}\right)+\ln^2\left(\frac{x_2}{y_2}\right)+\ln^2\left(\frac{x_2}{y_1}\right)-\ln^2\left(\frac{x_1}{y_2}\right)\right)\right\} \tag{A.1}$$

where:

$$x_{1,2} = \frac{m_1^2 + m_2^2 - m_3^2 \pm \sqrt{\Delta}}{2m_1^2} \tag{A.2}$$

$$y_{1,2} = \frac{m_1^2 - m_2^2 + m_3^2 \pm \sqrt{\Delta}}{2m_1^2} \tag{A.3}$$

$$\Delta = m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2 - i\ \mathrm{sign}(m_1^2)\ \epsilon \tag{A.4}$$

The formula (A.1) is valid in all kinematical regions because of the small imaginary part $i\ \epsilon$:

$$\sqrt{\Delta \pm i\epsilon} = \begin{cases} \sqrt{\Delta} \pm i\epsilon\,,\ \Delta \geq 0 \\ \pm i\sqrt{-\Delta}\,,\ \Delta \leq 0 \end{cases} \tag{A.5}$$

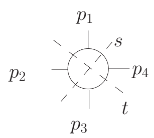
The two determinants are given by the relations:

$$det(G_{3m}) = m_1^2 m_2^2 - (m_1.m_2)^2 = -\frac{\Delta}{4} \tag{A.6}$$

$$det(S_{3m}) = 2m_1^2 m_2^2 m_3^2 \tag{A.7}$$

A.2 Four point functions

A.2.1 With zero mass



$$\begin{cases} s = s_{12} \\ t = s_{14} \\ u = s_{13} \end{cases}$$

$$I_4^n\left(s,t\right) = \frac{2}{st}\frac{r_\Gamma}{\epsilon^2}\left\{(-s)^{-\epsilon} + (-t)^{-\epsilon}\right\} - \frac{2}{st}F_0(s,t) \tag{A.8}$$

where:

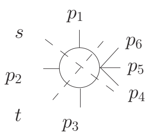
$$F_0(s,t) = \frac{1}{2}\left\{ln^2\left(\frac{s}{t}\right) + \pi^2\right\} \tag{A.9}$$

The determinants are given by:

$$det(G_0) = -2st(s+t) = 2stu \tag{A.10}$$

$$det(S_0) = (st)^2 = \langle 24342\rangle^2 \tag{A.11}$$

A.2.2 With one mass



$$\begin{cases} s = s_{12} \\ t = s_{23} \\ u = s_{13} \\ m^2 = s_{456} \end{cases}$$

$$I_4^n\left(s,t,m^2\right) = \frac{r_\Gamma}{st\epsilon^2}\left\{\left((-s)^{-\epsilon} + (-t)^{-\epsilon}\right) + \left((-s)^{-\epsilon} - (-m^2)^{-\epsilon}\right) + \left((-t)^{-\epsilon} - (-m^2)^{-\epsilon}\right)\right\} - \frac{2}{st}F_1\left(s,t,m^2\right) \tag{A.12}$$

where:

$$F_1\left(s,t,m^2\right) = Li_2\left(1 - \frac{m^2}{s}\right) + Li_2\left(1 - \frac{m^2}{t}\right) - Li_2\left(-\frac{s}{t}\right) - Li_2\left(-\frac{t}{s}\right) \tag{A.13}$$

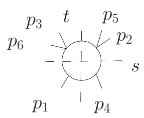
$$= F_0(s,t) + \left\{Li_2\left(1 - \frac{m^2}{s}\right) + Li_2\left(1 - \frac{m^2}{t}\right) - \frac{\pi^2}{3}\right\} \tag{A.14}$$

The determinants are given by:

$$det(G_1) = -2st\left(s + t - m^2\right) = 2stu \tag{A.15}$$

$$det(S_1) = (st)^2 = \langle 2m3m2\rangle^2 \tag{A.16}$$

A.2.3 With two adjacent masses



$$\begin{cases} s = s_{14} \\ t = s_{425} \\ m_1^2 = s_{25} \\ m_2^2 = s_{36} \end{cases}$$

$$I_4^n\left(s,t,m_1^2,m_2^2\right) = \frac{r_\Gamma}{(st)\epsilon^2}\left\{(-s)^{-\epsilon} + \left((-t)^{-\epsilon} - (-m_1^2)^{-\epsilon}\right) + \left((-t)^{-\epsilon} - (-m_2^2)^{-\epsilon}\right)\right\} - \frac{2}{st}F_{2A}\left(s,t,m_1^2,m_2^2\right) \tag{A.17}$$

where:

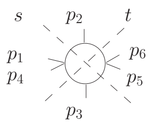
$$F_{2A}\left(s,t,m_1^2,m_2^2\right) = Li_2\left(1 - \frac{m_1^2}{t}\right) + Li_2\left(1 - \frac{m_2^2}{t}\right) + \frac{1}{2}ln\left(\frac{s}{t}\right)ln\left(\frac{m_2^2}{t}\right) + \frac{1}{2}ln\left(\frac{s}{m_2^2}\right)ln\left(\frac{m_1^2}{t}\right) \tag{A.18}$$

The determinants are given by:

$$det(G_{2A}) = -2s\left(m_1^2m_2^2 - t(m_1^2 + m_2^2 - s - t)\right) = -2s\langle 1m_14m_21\rangle \tag{A.19}$$

$$det(S_{2A}) = (st)^2 \tag{A.20}$$

A.2.4 With two opposite masses



$$\begin{cases} s = s_{143} \\ t = s_{243} \\ u = s_{23} \\ m_1^2 = s_{14} \\ m_2^2 = s_{56} \end{cases}$$

$$\begin{aligned} I_4^n\left(s,t,m_1^2,m_2^2\right) =& \frac{r_\Gamma}{(st-m_1^2m_2^2)\epsilon^2}\left\{\left((-s)^{-\epsilon}-(-m_1^2)^{-\epsilon}\right)+\left((-s)^{-\epsilon}-(-m_2^2)^{-\epsilon}\right)\right\} \\ &+\frac{r_\Gamma}{(st-m_1^2m_2^2)\epsilon^2}\left\{\left((-t)^{-\epsilon}-(-m_1^2)^{-\epsilon}\right)+\left((-t)^{-\epsilon}-(-m_2^2)^{-\epsilon}\right)\right\} \\ &-\frac{2}{st-m_1^2m_2^2}F_{2B}\left(s,t,m_1^2,m_2^2\right) \end{aligned} \tag{A.21}$$

where:

$$\begin{aligned} F_{2B}\left(s,t,m_1^2,m_2^2\right) =& -Li_2\left(1-\frac{m_1^2m_2^2}{st}\right)+Li_2\left(1-\frac{m_1^2}{s}\right) \\ &+Li_2\left(1-\frac{m_2^2}{s}\right)+Li_2\left(1-\frac{m_1^2}{t}\right)+Li_2\left(1-\frac{m_2^2}{t}\right)+\frac{1}{2}ln^2\left(\frac{s}{t}\right) \end{aligned} \tag{A.22}$$

$$= F_1\left(s,t,m_1^2\right)+F_1\left(s,t,m_2^2\right)-F_0(s,t)-\left\{Li_2\left(1-\frac{m_1^2m_2^2}{st}-\frac{\pi^2}{6}\right)\right\} \tag{A.23}$$

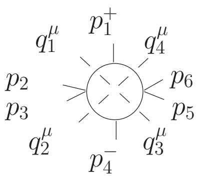
The determinants are given by:

$$det(G_{2B}) = -2\left(m_1^2m_2^2-st\right)\left(m_1^2+m_2^2-s-t\right) = 2u\left(st-m_1^2m_2^2\right) \tag{A.24}$$

$$det(S_{2B}) = \left(st-m_1^2m_2^2\right)^2 = \langle 2m_1 3m_1 2\rangle^2 = \langle 2m_2 3m_2 2\rangle^2 \tag{A.25}$$

B. Proof of rule (2)

In this appendix, we want to prove that, in the case of the box with two opposite masses, the helicities of the two opposite massless legs must be the same otherwise the coefficient in front is zero. To do that, we consider the following box integrals where the helicities of the two opposite massless legs is different.



$$\begin{cases} \forall i \in [1\ldots 6],\ p_i^2 = 0 \\ p_{23} = p_2 + p_3 \\ p_{56} = p_5 + p_6 \end{cases}$$

We assume that the helicity of the photon p_1 is positive and the helicity of the photon p_4 is negative. Using the four cuts techniques, the coefficient, called C, in front of this master integral is given by, in scalar QED:

$$C \propto \sum_{i=a,b} \varepsilon_1^+.q_{1i}\varepsilon_2^-.q_{2i} \tag{B.1}$$

where q_{1i} and q_{2i} are the solution of the four cuts conditions:

$$\delta(q_1^2) = 0 \tag{B.2}$$
$$\delta(q_2^2) = 0 \tag{B.3}$$
$$\delta(q_3^2) = 0 \tag{B.4}$$
$$\delta(q_4^2) = 0 \tag{B.5}$$

So first we solve this system and after we will calculate (B.1).

We choose as a base of the four-dimension Minkowski space: $B = \{p_1^\mu, p_4^\mu, \langle 1\gamma^\mu 4\rangle, \langle 4\gamma^\mu 1\rangle\}$. In our case, q_1^μ can be taken as a four-dimension vector, therefore, we can project it on the base B:

$$q_1^\mu = a\ p_1^\mu + b\ p_4^\mu + \frac{c}{2}\langle 1\gamma^\mu 4\rangle + \frac{d}{2}\langle 4\gamma^\mu 1\rangle \tag{B.6}$$

So to know the vector q_1^μ, we have to calculate, the four coefficient a, b, c and d. The conditions (B.2) and (B.5) impose:

$$(q_1 - p_1)^2 = 0 \Leftrightarrow 2\,(p_1.q_1) = 0 \Leftrightarrow b = 0 \tag{B.7}$$

The conditions (B.3) and (B.4) impose:

$$(q_1 + p_{23} + p_4)^2 = 0 \Leftrightarrow 2p_4.\,(q_1 + p_{23}) = 0 \Leftrightarrow a s_{14} + 2(p_{23}.p_4) = 0 \Leftrightarrow a = -\frac{2(p_{23}.p_4)}{s_{14}} \tag{B.8}$$

The first condition (B.2), knowing $b = 0$, imposes:

$$q_1^2 = 0 \Leftrightarrow c\ d = 0 \Leftrightarrow c = 0 \text{ or } d = 0 \tag{B.9}$$

If we assume that $d = 0$, therefore the conditions (B.2) and (B.3) impose that:

$$(q_1 + p_{23})^2 = 0 \Leftrightarrow 2(p_{23}.q_1) = -s_{23} \Leftrightarrow c = \frac{\langle 4 p_{23} 1\rangle}{s_{14}} \tag{B.10}$$

and finally we obtain $q_1^\mu = -\frac{\langle 4P_{23}\gamma^\mu 14\rangle}{2s_{14}}$.
Else if we assume $c = 0$, therefore the conditions (B.2) and (B.3) impose that:

$$(q_1 + p_{23})^2 = 0 \Leftrightarrow 2(p_{23}.q_1) = -s_{23} \Leftrightarrow d = \frac{\langle 1 p_{23} 4\rangle}{s_{14}} \tag{B.11}$$

and in this case we obtain $q_1^\mu = -\frac{[4P_{23}\gamma^\mu 14]}{2s_{14}}$. Finally according to the four cuts techniques, the loop momenta is found to be:

$$\begin{cases} q^\mu_{1\,a} = -\frac{\langle 4P_{23}\gamma^\mu 14\rangle}{2s_{14}} \\ q^\mu_{1\,b} = -\frac{[4P_{23}\gamma^\mu 14]}{2s_{14}} \end{cases} \tag{B.12}$$

From the formula (B.12), we can compute $q_{2a/b}$ and we obtain:

$$\begin{cases} q^\mu_{2\,a} = q^\mu_{1\,a} + p^\mu_{23} = \frac{\langle 4\gamma^\mu P_{23} 14\rangle}{2s_{14}} \\ q^\mu_{2\,b} = q^\mu_{1\,b} + p^\mu_{23} = \frac{[4\gamma^\mu P_{23} 14]}{2s_{14}} \end{cases} \tag{B.13}$$

We are now ready to compute the left hand side of equation (B.1) inserting the formula (B.12) and (B.13), we obtain directly that:

$$C \propto \varepsilon_1^+ . q_{1a} \varepsilon_2^- . q_{2a} + \varepsilon_1^+ . q_{1b} \varepsilon_2^- . q_{2b} = 0 \tag{B.14}$$

Therefore the hypothesis that the two photons p_1 and p_4 have two different helicities implies that the coefficient in front of the two opposite mass integrals is zero.

References

[1] R. Karplus and M. Neumann, *Non-linear interactions between electromagnetic fields*, *Phys. Rev.* **80** (1950) 380;*The scattering of light by light*, *Phys. Rev.* **83** (1951) 776;
B. De Tollis, *Dispersive approach to photon-photon scattering*, *Nuovo Cim.* **32** (1964) 757; *The scattering of photons by photons*, *Nuovo Cim.* **35** (1965) 1182.

[2] Z. Bern, A. De Freitas, L.J. Dixon, A. Ghinculov and H.L. Wong, *QCD and QED corrections to light-by-light scattering*, *JHEP* **11** (2001) 031 [hep-ph/0109079].

[3] T. Binoth, E.W.N. Glover, P. Marquard and J.J. van der Bij, *Two-loop corrections to light-by-light scattering in supersymmetric QED*, *JHEP* **05** (2002) 060 [hep-ph/0202266].

[4] G. Mahlon, *Use of recursion relations to compute one loop helicity amplitudes*, hep-ph/9412350.

[5] Z. Nagy and D.E. Soper, *Numerical integration of one-loop Feynman diagrams for n-photon amplitudes*, *Phys. Rev.* **D 74** (2006) 093006 [hep-ph/0610028].

[6] G. Ossola, C.G. Papadopoulos and R. Pittau, *Numerical evaluation of six-photon amplitudes*, *JHEP* **07** (2007) 085 [arXiv:0704.1271].

[7] T. Binoth, G. Heinrich, T. Gehrmann and P. Mastrolia, *Six-photon amplitudes*, *Phys. Lett.* **B 649** (2007) 422 [hep-ph/0703311].

[8] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One loop n point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys.* **B 425** (1994) 217 [hep-ph/9403226].

[9] R. Britto, F. Cachazo and B. Feng, *Generalized unitarity and one-loop amplitudes in $N = 4$ super-Yang-Mills*, *Nucl. Phys.* **B 725** (2005) 275 [hep-th/0412103].

[10] Z. Bern, L. Dixon and D.A. Kosower, *One-loop amplitudes for e^+e^- to four partons*, hep-ph/9708239.

[11] R. Britto, E. Buchbinder, F. Cachazo and B. Feng, *One-loop amplitudes of gluons in SQCD*, *Phys. Rev.* **D 72** (2005) 065012 [hep-ph/0503132].

[12] R. Britto, B. Feng and P. Mastrolia, *The cut-constructible part of QCD amplitudes*, *Phys. Rev.* **D 73** (2006) 105004 [hep-ph/0602178].

[13] P. Mastrolia, *On triple-cut of scattering amplitudes*, *Phys. Lett.* **B 644** (2007) 272 [hep-th/0611091].

[14] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, *D-dimensional unitarity cut method*, *Phys. Lett.* **B 645** (2007) 213 [hep-ph/0609191]; *Unitarity cuts and reduction to master integrals in d dimensions for one-loop amplitudes*, *JHEP* **03** (2007) 111 [hep-ph/0612277].

[15] R. Britto and B. Feng, *Unitarity cuts with massive propagators and algebraic expressions for coefficients*, *Phys. Rev.* **D 75** (2007) 105006 [hep-ph/0612089].

[16] L.J. Dixon, *Calculating scattering amplitudes efficiently*, hep-ph/9601359.

[17] Z. Xu, D.H Zhang and L. Chang, *Helicity amplitudes for multiple bremmstralung in massless non-abelian gauge theories*, *Nucl. Phys.* **B 291** (1987) 392.

[18] W.L. van Neerven and J.A.M. Vermaseren, *Large loop integrals*, *Phys. Lett.* **B 137** (1984) 241.

[19] Z. Bern, L.J. Dixon and D.A. Kosower, *Dimensionally regulated one loop integrals*, *Phys. Lett.* **B 302** (1993) 299 [*Erratum ibid.* **B 318** (1993) 649] [hep-ph/9212308]; *Dimensionally regulated pentagon integrals*, *Nucl. Phys.* **B 412** (1994) 751 [hep-ph/9306240].

[20] T. Binoth, J.P. Guillet, G. Heinrich, E. Pilon and C. Schubert, *An algebraic/numerical formalism for one-loop multi-leg amplitudes*, *JHEP* **10** (2005) 015 [hep-ph/0504267].

[21] T. Binoth, J.P. Guillet, G. Heinrich and C. Schubert, *Calculation of* 1*-loop hexagon amplitudes in the Yukawa model*, *Nucl. Phys.* **B 615** (2001) 385 [hep-ph/0106243].

[22] T. Binoth, J.P. Guillet and G. Heinrich, *Algebraic evaluation of rational polynomials in one-loop amplitudes*, *JHEP* **02** (2007) 013 [hep-ph/0609054].

[23] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *Fusing gauge theory tree amplitudes into loop amplitudes*, *Nucl. Phys.* **B 435** (1995) 59 [hep-ph/9409265].

[24] D. Forde, *Direct extraction of one-loop integral coefficients*, *Phys. Rev.* **D 75** (2007) 125019 [arXiv:0704.1835].

[25] C. Bernicot, F. Boudjema, J.P. Guillet, N. Le Duc and E. Pilon, *Landau singularities*, to appear.

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